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On the Relations Between Polarized Structure Functions

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The status of the twist–2 and the twist–3 integral relations between polarized structure functions in deep inelastic scattering for electromagnetic and electroweak interactions is reviewed. One novel integral relation for the twist–3 contributions can be tested in the upcoming high statistics measurements of the structure function $g_2(x,Q^2)$ in the range $Q^2 \gtrsim M_p^2$.

§1. Introduction

The measurement of the nucleon structure functions in polarized deeply inelastic scattering reveals the behaviour of quarks and gluons in outer magnetic fields and allows to test basic predictions of Quantum Chromodynamics (QCD) in the short–distance regime $Q^2 \gg M_p^2$. In this domain the structure of the hadronic tensor $W_{\mu\nu}$ which describes the process can be investigated in terms of the light–cone expansion [1)]. We restrict our consideration to those contributions to $W_{\mu\nu}$ which contribute to the scattering cross sections in the massless quark case. There $W_{\mu\nu}$ is a current conserving quantity for the electro–weak interactions,

$$
\begin{aligned}
W_{\mu\nu} =& \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x,Q^2) + \frac{\hat{P}_\mu \hat{P}_\nu}{P.q} F_2(x,Q^2) - i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda P^\sigma}{2P.q} F_3(x,Q^2) \\
& + i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda S^\sigma}{P.q} g_1(x,Q^2) + i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda (P.q S^\sigma - S.q P^\sigma)}{(P.q)^2} g_2(x,Q^2) \\
& + \left[\frac{\hat{P}_\mu \hat{S}_\nu + \hat{S}_\mu \hat{P}_\mu}{2} - S.q \frac{\hat{P}_\mu \hat{P}_\nu}{P.q}\right] \frac{g_3(x,Q^2)}{P.q} \\
& + S.q \frac{\hat{P}_\mu \hat{P}_\nu}{(P.q)^2} g_4(x,Q^2) + \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \frac{(S.q)}{P.q} g_5(x,Q^2), \qquad (1\cdot 1)
\end{aligned}
$$

with

$$
\hat{P}_\mu = P_\mu - \frac{P.q}{q^2} q_\mu, \qquad \hat{S}_\mu = S_\mu - \frac{S.q}{q^2} q_\mu \ .
$$

Here P, q and S are the 4–vectors of the nucleon momentum, the momentum transfer and the nucleon spin, respectively. The unpolarized structure functions are denoted by $F_i(x,Q^2)$ and the polarized structure functions by $g_i(x,Q^2)$. $W_{\mu\nu}$ refers to the contributions per current, where the propagator terms are separated off. In the case of electromagnetic interactions only the structure functions F_1, F_2, g_1 and g_2

contribute. The structure functions aquire their Q^2 dependence as scaling violations due to higher order QCD corrections and power corrections, such as target mass corrections and dynamical higher twist terms. The latter contributions turn out to be particularly essential, even if QCD corrections are not yet considered, in the range of lower values of Q^2, $M_p^2 \lesssim Q^2$, where most of the present data are taken.

§2. Twist Decomposition

The Fourier transform of the Compton amplitude $T_{\mu\nu}^{NC}(x)$ for neutral current interactions reads

$$\begin{aligned} i\int d^4x e^{iqx}\widehat{T}_{\mu\nu}^{NC} &= -\int \frac{d^4k}{(2\pi)^4}\overline{U}(k)\gamma_\mu(g_{V_1}+g_{A_1}\gamma_5)\frac{\hat{k}+\hat{q}}{(k+q)^2}\gamma_\nu(g_{V_2}+g_{A_2}\gamma_5)U(k) \\ &\quad -\int \frac{d^4k}{(2\pi)^4}\overline{U}(k)\gamma_\mu(g_{V_1}+g_{A_1}\gamma_5)\frac{\hat{k}-\hat{q}}{(k-q)^2}\gamma_\nu(g_{V_2}+g_{A_2}\gamma_5)U(k), \\ &= -i(g_{V_1}g_{V_2}+g_{A_1}g_{A_2})\varepsilon_{\mu\alpha\nu\beta}q^\alpha u_+^\beta \\ &\quad +(g_{V_1}g_{A_2}+g_{A_1}g_{V_2})S_{\mu\alpha\nu\beta}[q^\alpha u_-^\beta + u^{\alpha\beta}], \end{aligned} \tag{2·1}$$

where $U(k)=\int d^4k/(2\pi)^4 e^{-ikx}\psi(x)$, $S_{\mu\alpha\nu\beta}=g_{\mu\alpha}g_{\mu\beta}+g_{\mu\beta}g_{\nu\alpha}-g_{\mu\nu}g_{\alpha\beta}$, and

$$u_\pm^\beta = -\int \frac{d^4k}{(2\pi)^4}\overline{U}(k)\frac{\gamma_\beta\gamma_5}{(k+q)^2}U(k) \ \mp\ (q\leftrightarrow -q)\ , \tag{2·2}$$

$$u^{\alpha\beta} = -\int \frac{d^4k}{(2\pi)^4}\overline{U}(k)\frac{k_\alpha\gamma_\beta\gamma_5}{(k+q)^2}U(k) \ -\ (q\leftrightarrow -q)\ . \tag{2·3}$$

The expansion of the denominators $(k\pm q)^2$ results into the operator–representation

$$u_\pm^\beta = \sum_{n\ even,odd}\left(\frac{2}{Q^2}\right)^{n+1} q_{\mu_1}\dots q_{\mu_n}\Theta^{+\beta\{\mu_1\cdots\mu_n\}}\ , \tag{2·4}$$

$$u^{\alpha\beta} = \sum_{n\ even}\left(\frac{2}{Q^2}\right)^{n+1} q_{\mu_1}\dots q_{\mu_n}\Theta^{+\beta\{\alpha\mu_1\cdots\mu_n\}}\ . \tag{2·5}$$

The operators $\Theta^{+\beta\{\mu_1\cdots\mu_n\}}$ are given by

$$\Theta^{+\beta\{\mu_1\cdots\mu_n\}} = \int\frac{d^4k}{(2\pi)^4}\overline{U}(k)\gamma_\beta\gamma_5 k_{\mu_1}\dots k_{\mu_n}U(k) = \Theta_S^{+\beta\{\mu_1\cdots\mu_n\}} + \Theta_R^{+\beta\{\mu_1\cdots\mu_n\}} \tag{2·6}$$

and may be decomposed into a fully symmetric and a remainder part with respect to their indices. The former contribution is of twist–2 while the latter is a twist–3 operator. In general this decomposition has to be performed accounting for target mass effects, see Ref. [2] for details. In the massless case the corresponding representations were given in Ref. [3]*.

* For the twist–2 contributions one may obtain representations using the covariant parton model[4] in the *massless* case and lowest order in α_s.

Whereas the above decomposition of the Compton amplitude yields contributions to all the eight Lorentz tensors of Eq. (1) due to $\Theta_S^{+\beta\{\mu_1\cdots\mu_n\}}$ this is the case for the polarized part for $\Theta_R^{+\beta\{\mu_1\cdots\mu_n\}}$ only keeping all the nucleon mass terms [2]. Taking the limit $M_p \to 0$ only contributions $\propto g_2$ and g_3 are obtained, which were studied in Ref. [3] previously.

The deep inelastic structure functions contain contributions of different twist. Since the individual twist terms obey independent renormalization group equations their scaling violations are different. Moreover the different twist operators aquire separate expectation values which are related to different target mass corrections in general. Because of this the structure functions F_i and g_i have to be represented as linear superpositions of their twist contributions, the scaling violations of which are calculated individually. Henceforth we will discuss the twist contributions separately.

§3. Twist–2 Relations

In lowest order in α_s the twist–2 contributions to the structure functions $g_i|_{i=1}^5$ are determined by a single quarkonic operator matrix element a_n for each moment n in the massless case

$$g_i(n) = \int_0^1 dx x^{n-1} g_i(x) \ . \tag{3·1}$$

These non–perturbative functions are different for the sets of structure functions $g_{1,2}$ and $g_{3,4,5}$ due to the corresponding quark contents. At the level of twist–2 the former ones are $\propto \Delta q(x) + \Delta\overline{q}(x)$ while the latter are of the type $\Delta q(x) - \Delta\overline{q}(x)$.

The nucleon mass dependence may induce involved expressions for the different structure functions which are typically of the form, cf. [2],*

$$\begin{aligned} g_{3\ \tau=2}^{\pm}(x) = \sum_q g_V^q g_A^q \Biggl\{ & \frac{2x^2}{\xi(1+4M^2x^2/Q^2)^{3/2}} \int_\xi^1 \frac{d\xi_1}{\xi_1} F^{\pm q}(\xi_1) \\ & + \frac{12M^2x^3/Q^2}{(1+4M^2x^2/Q^2)^2} \int_\xi^1 \frac{d\xi_1}{\xi_1} \int_{\xi_1}^1 \frac{d\xi_2}{\xi_2} F^{\pm q}(\xi_2) \\ & + \frac{3}{2} \frac{(4M^2x^2/Q^2)^2}{(1+4M^2x^2/Q^2)^{5/2}} \int_\xi^1 d\xi_1 \int_{\xi_1}^1 \frac{d\xi_2}{\xi_2} \int_{\xi_2}^1 \frac{d\xi_3}{\xi_3} F^{\pm q}(\xi_3) \Biggr\} \ . \end{aligned} \tag{3·2}$$

Despite of this one may show, however, that the relations

$$g_2^{\rm II}(x,Q^2) = -g_1^{\rm II}(x,Q^2) + \int_x^1 \frac{dy}{y} g_1^{\rm II}(y,Q^2) \tag{3·3}$$

$$g_3^{\rm II}(x,Q^2) = 2x \int_x^1 \frac{dy}{y^2} g_4^{\rm II}(y,Q^2) \tag{3·4}$$

hold *in general*[2] **. In this way the target mass corrections are absorbed, without

* Numerical results are presented in Ref. [5].

** The validity of Eq. (3·3) in the presence of target mass corrections was shown in [6] before. Expansions in terms of M^2/Q^2 as considered in this paper, however, turn out to introduce artificial singularities in the range $x \to 1$, which are not present in the resummed expressions [2].

changing the form of the Wandzura–Wilczek relation (3·3) [7] and a relation by the author and Kochelev [3), 4)] (3·4) in the massless case. As was shown in Ref. [2] the Wandzura–Wilczek relation also holds in the case that the quark mass terms are considered as well.

The relation

$$g_4^{\rm II}(x,Q^2) = 2xg_5^{\rm II}(x,Q^2) + \Delta(M^2/Q^2) \tag{3·5}$$

with $\lim_{\rho\rightarrow 0}\Delta(\rho) = 0$ turns out to be a relation between structure functions only in the massless case, where it was found by Dicus [8]. In the presence of target masses it is modified [9] in the same way as the Callan–Gross relation [10].

§4. Why do integral relations occur?

The Callen–Gross or Dicus relation and the Wandzura–Wilczek relation are structurally different. Whereas the first one is a purely algebraic relation the second one contains an integral term. The structure function under the integral $g_1^{\rm II}(x,Q^2)$ has a partonic interpretation as well as the twist–2 contributions to the unpolarized structure function $F_1^{\rm II}(x,Q^2)$. This difference can be understood studying firstly deeply–virtual non–forward Compton scattering $\gamma_1^*(q_1)+p_1 \rightarrow \gamma_2^*(q_2)+p_2$ and performing then the limit to the case of forward scattering in the Compton amplitude [11]

$$\lim_{p_-\rightarrow 0} T_{\mu\nu}(q,p_+,p_-) = T_{\mu\nu}^{\rm forw.}(p,q)~, \tag{4·1}$$

with $p_\pm = p_2 \pm p_1, p = p_+/2$ and $q = (q_1+q_2)/2$. The Compton amplitude has the representation

$$T_{\mu\nu}(q,p_+,p_-) = T_{\mu\nu}^{\rm sym}(q,p_+,p_-) + T_{\mu\nu}^{\rm asym}(q,p_+,p_-)~. \tag{4·2}$$

From the non–forward expressions [11] one obtains in the forward limit :

$$\begin{aligned} T_{\mu\nu}^{\rm sym}(q,p) = &\left(g_{\mu\nu} - \frac{p_\mu q_\nu + p_\nu q_\mu}{p.q}\right)\int Dz \\ &\left\{\left(\frac{1}{\xi+z_+-i\varepsilon} - \frac{1}{\xi-z_+-i\varepsilon}\right)\right. \\ &\left. -z_+\left[\frac{1}{(\xi+z_+-i\varepsilon)^2} - \frac{1}{(\xi-z_+-i\varepsilon)^2}\right]\right\}F(z_+,z_-)~. \end{aligned} \tag{4·3}$$

Here $F(z_+,z_-)$ denotes a distribution amplitude and $\xi = -q^2/2p.q$. z_+ and z_- are momentum fractions with

$$Dz = dz_+dz_-\theta(1+z_++z_-)\theta(1+z_+-z_-)\theta(1-z_++z_-)\theta(1-z_+-z_-) \tag{4·4}$$

The integral over z_- can be performed analytically. Due to the structure of the function $F(z_+,z_-)$ the representation

$$\int_{-1+|z_+|}^{+1-|z_+|} dz_- F(z_+,z_-) = \int_{z_+}^{{\rm sign}(z_+)} \frac{dz}{z} f(z) \tag{4·5}$$

holds [11]. Here $f(z)$ denotes the number density of quarks or antiquarks. Evidently the structure functions in Eq. (4·3) corresponding to the two tensors are equal but may contain integral terms w.r.t. a *partonic* function. Evaluating the integral

$$\int_{-1}^{+1} dz_+ \frac{z_+}{(\xi \pm z_+ - i\varepsilon)^2} \int_{z_+}^{\mathrm{sign}(z_+)} \frac{dz}{z} f(z)$$
$$= \qquad \mp \int_{-1}^{+1} dz_+ \frac{1}{\xi \pm z_+ - i\varepsilon} \left[\int_{z_+}^{\mathrm{sign}(z_+)} \frac{dz}{z} f(z) - f(z)\right] \qquad (4\cdot6)$$

the integral terms contributing to the first and second propagator term in Eq. (4·3) cancel, while the algebraic term remains.

For the polarized contribution this cancellation does not occur :

$$T_{\mu\nu}^{\mathrm{asym}}(q,p) = i\varepsilon_{\mu\nu}^{\gamma\beta} \frac{q_\gamma p_\beta}{(p.q)^2} q.S \qquad (4\cdot7)$$
$$\times \int_{-1}^{+1} dz_+ \left[\frac{1}{\xi + z_+ - i\varepsilon} + \frac{1}{\xi - z_+ - i\varepsilon}\right] \left[\int_{z_+}^{\mathrm{sign}(z_+)} \frac{dz}{z} f_5(z) - f_5(z)\right]$$
$$- i\varepsilon_{\mu\nu}^{\gamma\beta} \frac{q_\gamma S_\beta}{p.q} \int_{-1}^{+1} dz_+ \left[\frac{1}{\xi + z_+ - i\varepsilon} + \frac{1}{\xi - z_+ - i\varepsilon}\right] \int_{z_+}^{\mathrm{sign}(z_+)} \frac{dz}{z} f_5(z) \ .$$

Rewriting the tensors in terms of those defining the structure functions $g_1(x,Q^2)$ and $g_2(x,Q^2)$, Eq. (1·1), the Wandzura–Wilczek relation is obtained. The occurrence of integral contributions or purely algebraic terms in relations between deeply inelastic structure functions or other hard scattering processes has thus to be considered as the regular case, with the possibility that these terms may cancel.

§5. Twist–3 Relations

A unique picture for the twist–3 terms can only be obtained including the target mass corrections, cf. [2]. In this case all polarized structure functions contain twist–3 contributions. Indeed the consideration of mass corrections appears to be necessary also for consistency reasons since the structure functions containing twist–3 terms in the case of longitudinal nucleon polarization are weighted by a factor of M^2/S in the scattering cross section. The explicit relations for the target mass corrections to the twist–3 contributions to $g_i|_{i=1}^5$ are given in Ref. [2].

They obey the following relations :

$$g_1^{\mathrm{III}}(x,Q^2) = \frac{4M^2x^2}{Q^2}\left[g_2^{\mathrm{III}}(x,Q^2) - 2\int_x^1 \frac{dy}{y} g_2^{\mathrm{III}}(y,Q^2)\right] \ , \qquad (5\cdot1)$$

$$\frac{4M^2x^2}{Q^2} g_3^{\mathrm{III}}(x,Q^2) = g_4^{\mathrm{III}}(x,Q^2)\left(1 + \frac{4M^2x^2}{Q^2}\right) + 3\int_x^1 \frac{dy}{y} g_4^{\mathrm{III}}(y,Q^2) \ , \qquad (5\cdot2)$$

$$2x g_5^{\mathrm{III}}(x,Q^2) = -\int_x^1 \frac{dy}{y} g_4^{\mathrm{III}}(y,Q^2) \ , \qquad (5\cdot3)$$

which hold after the inclusion of the target mass corrections to all orders in $(M^2/Q^2)^k$. Whereas the relations (5·2,5·3) are relevant in the presence of weak interactions only,

Eq. (5·1) can be tested already for purely electromagnetic interactions in the domain of lower values of Q^2.

For an experimental determination of the twist–3 contributions to the structure functions $g_1(x,Q^2)$ and $g_2(x,Q^2)$ one can proceed as follows. From Eq. (5·1) it is evident, that for $Q^2 \gg M^2$ $g_1(x,Q^2)$ receives only twist–2 contributions. g_1 can be measured firstly in this range. Its twist–2 contribution at lower values of Q^2 can be obtained solving the twist–2 evolution equations for $g_1(x,Q^2)$. Then one can determine the twist–2 contribution to $g_2(x,Q^2)$ by the Wandzura–Wilczek relation, Eq. (3·3). Assuming that the contributions of twist–4 and higher are suppressed relatively to the the twist–3 contributions to $g_{1,2}(x,Q^2)$ the latter can be extracted form the data by subtraction of the twist–2 pieces. Relation (5·1) [2] may now be tested by calculating the twist–3 contribution to $g_1(x,Q^2)$ from the twist–3 contribution to $g_2(x,Q^2)$ and comparing with the measurement.

Finally we would like to comment on two other relations. The Burkhardt–Cottingham sum rule [12]

$$\int_0^1 dx g_2(x,Q^2) = 0 \tag{5·4}$$

is consistent with the results of the local light cone expansion. Both the corresponding expectation values for twist–2 and twist–3 are absent in the respective series expansion also in the presence of target mass corrections [2].

A second relation [13]

$$\int_0^1 dx x \left[g_1(x) + 2g_2(x)\right]_{\rm valence} = 0 \tag{5·5}$$

also holds in the presence of target mass corrections as long as $Q^2 > M_p^2$. One may cast the respective relations into the following form, cf. [2]:

$$\int_0^1 dx x \left[g_1(x) + 2g_2(x)\right]_{\rm valence} = \sum_q \frac{e_q^2}{2}\frac{m_q}{M_p}\int_0^1 dx \frac{h_1^q(x) - \overline{h}_1^q(x)}{\left(1 - \dfrac{M_p^2 x^2}{Q^2}\right)^2}\ . \tag{5·6}$$

Here we allowed for finite quark masses m_q and $h_1^q(x)$ denotes the transversity distribution of the quark q. The integral in Eq. (5·6) is finite under the above condition and one may perform the limit $m_q \to 0$ to obtain Eq. (5·5).

§6. Summary

In the range of low values of $Q^2 \gtrsim M_p^2$ nucleon mass corrections to the polarized structure functions in deep inelastic scattering are essential. After the inclusion of these corrections a symmetric picture is obtained comparing the respective twist–2 and twist–3 terms, unlike the massless case [3]. In lowest order in α_s the twist–2 and the twist–3 contributions of the five polarized structure functions are connected by three relations. These are the Wandzura–Wilczek relation, a relation by the author and Kochelev and the Dicus relation for the twist–2 terms, and three relations by

the author and Tkabladze for the twist–3 terms. While the Dicus relation receives a finite target mass correction, the other relations do not, or they do firstly result after the inclusion of the target mass corrections at all. The relations being present for purely electromagnetic interactions can be tested through precision measurements of the structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$ in the near future.

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